Combinatorics in Banach space theory (MIM UW 2014/15)

PROBLEMS (Part 4)

Notation: X_{GM} = the Gowers–Maurey space, X_{S} = the Schlumprecht space; by $\|\cdot\|_{\mathsf{GM}}$ and $\|\cdot\|_{\mathsf{S}}$ we denote their norms; $\mathcal{F}$ = the class of functions considered in the construction of both of these spaces; $(e_n)_{n=1}^{\infty}$ = the canonical basis of c_{00}.

PROBLEM 4.1. Assume that a Banach space X admits an asymptotic biorthogonal system with constant $\delta \in (0, \frac{1}{2})$. Show that X is $\frac{1}{2\delta}$-distortable, and hence if X contains asymptotic biorthogonal systems with arbitrarily small constants, then X is arbitrarily distortable.

Hint. It is enough to use only three sets: A_1, A_2 and A_1^* from the given asymptotic biorthogonal system $(A_n)_{n=1}^{\infty} \subset S_X$, $(A_n^*)_{n=1}^{\infty} \subset B_{X^*}$. Can you reformulate the argument which we used to prove that Tsirelson's space $\mathcal{T}$ is $(2-\varepsilon)$-distortable, and explain how these sets A_1, A_2 and A_1^* may look like in this situation?

PROBLEM 4.2. Prove that

$$\Big\| \sum_{j=1}^{n} e_j \Big\|_{\mathsf{S}} = \frac{n}{\log_2(n+1)} \quad \text{for every } n \in \mathbb{N}.$$

PROBLEM 4.3. Show that X_{GM} is reflexive.

Hint. Verify that the canonical basis is boundedly complete and shrinking.

PROBLEM 4.4. Show that no Banach space satisfying a lower f-estimate, for some $f \in \mathcal{F}$, can be renormed in a uniformly convex way.

Hint. Such a space must contain ℓ_1^n's uniformly.

Remark. In particular, X_{GM} does not admit any uniformly convex renorming (in other words, is not superreflexive). A construction of a uniformly convex HI space was given in the paper [V. Ferenczi, *A uniformly convex hereditarily indecomposable Banach space*, Israel J. Math. 102 (1997), 199–225].

PROBLEM 4.5. Show that for every Banach space X the following assertions are equivalent:

(i) X is HI;
(ii) for any infinite-dimensional closed subspaces Y and Z of X, the distance between the unit spheres of Y and Z is zero;
(iii) for any infinite-dimensional closed subspaces Y and Z of X, and every $\delta > 0$, there exist vectors $y \in Y$ and $z \in Z$ such that $\delta\|y+z\| > \|y-z\|$;
(iv) for every infinite-dimensional closed subspace Y of X and every set $W \subset B_{X^*}$ which is ε-norming for Y, with some $\varepsilon > 0$ (that is, $\sup_{\varphi\in W} |\varphi(y)| \geqslant \varepsilon\|y\|$ for every $y \in Y$), the preannihilator ${}^{\perp}W = \{x \in X \colon \varphi(x) = 0 \text{ for each } \varphi \in W\}$ is finite-dimensional.

PROBLEM 4.6. Let X and Y be Banach spaces. We call a bounded linear operator $T \in \mathscr{B}(X,Y)$ *infinitely singular* provided that for each $\varepsilon > 0$ there exists an infinite-dimensional subspace Z of X such that $\|T|_Z\| < \varepsilon$. Prove that if T is not infinitely singular, then the restriction of T to some finite-codimensional subspace of X is bounded below (an isomorphism onto its range) and, moreover, the complementary subspace can be taken to be $\ker T$.

PROBLEM 4.7. Let X be a complex Banach space and let $T \in \mathscr{B}(X)$. We call a number $\lambda \in \mathbb{C}$ *infinitely singuar for* T if $T - \lambda I_X$ is infinitely singular. Assuming X is HI prove that:

(a) there exists at most one complex number that is infinitely singular for T (hence, there is exactly one such number unless $\dim X < \infty$);
(b) if λ is infinitely singular for T, then $T - \lambda I_X$ is strictly singular (i.e., it is not bounded below on any infinite-dimensional subspace of X).

PROBLEM 4.8. Let X be a complex Banach space and $T \in \mathscr{B}(X)$. We denote by F_T the set of all complex numbers that are not infinitely singular for T. Prove that if $\lambda \in \partial\sigma(T) \cap F_T$, then λ is an isolated point of $\sigma(T)$.

Remark. This assertion is the key step in proving that $F_T \neq \mathbb{C}$ whenever X is infinite-dimensional.

PROBLEM 4.9. Let X be a real HI space and let $T \in \mathscr{B}(X)$. Let also $\widetilde{T}$ be the natural extension of T to the complexification space $X_{\mathbb{C}}$ of X. Prove that either $T - \lambda I_X$ is strictly singular for some $\lambda \in \mathbb{R}$, or $T^2 - 2\mathrm{Re}\,\lambda T + |\lambda|^2 I_X$ is strictly singular for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Prove also that $\sigma(\widetilde{T})$ is invariant under complex conjugation and the set $\sigma(\widetilde{T}) \cap \{z\colon \mathrm{Im}\, z \geqslant 0\}$ is finite or consists of a convergent sequence with its limit.

Next, explain how these assertions imply that every bounded linear operator on a real HI space is either strictly singular or Fredholm with index 0.

PROBLEM 4.10. Show that all closed hyperplanes (subspaces of codimension 1) of any Banach space are mutually isomorphic.

PROBLEM 4.11. Prove that every HI Banach space embeds isomorphically into ℓ_∞.

Hint. Exploit the fact that in the dual of any HI space X we have a countable set which separates points and is, say, $\frac{1}{2}$-norming for any given separable subspace X. This follows (how?) from the characterization given in Problem 4.5(iv). Next, use Problems 4.6 and 4.7.

PROBLEM 4.12. Without using any operator-theoretic tools (in particular, without using the knowledge about the form of operators on an HI space) show that the basic sequences $(e_n)_{n=1}^\infty$ and $(e_n)_{n=2}^\infty$ are not equivalent in X_{GM}.

Remark. You are supposed to prove this more or less directly from the definition of the norm $\|\cdot\|_{\mathsf{GM}}$, however, some deep estimates (like the one for the sum a R.I.S. of length $N \in L$) will be quite indispensable.